Vandermonde determinant with higher degree

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The evaluation of Vandermonde determinant, that is,

$$V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ : & : & : & & : \\ : & : & : & & : \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}_{(n)} = \prod_{1 \le p < q \le n} \left(x_q - x_p \right) \qquad (1)$$

is a good exercise for beginners in studying the theory of determinants. The readers are advised to carry out the necessary calculations, employing the usual techniques such as Laplace expansion or mathematical induction, before reading this article.

This passage aims to define a higher degree Vandermonde determinant, to evaluate it and to apply the result in considering the existence of non-trivial solutions in certain kind of homogeneous symmetric system of equations.

It is natural that the Vandermonde determinant can be extended as follows :

$$V_n^{(i)} = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{i-1} & x_1^{i+1} & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{i-1} & x_2^{n-1} & \dots & x_2^n \\ : & : & : & & : & : & & : \\ : & : & : & & : & : & & : \\ 1 & x_n & x_n^2 & \dots & x_n^{i-1} & x_n^{i+1} & \dots & x_n^n \end{vmatrix}_{(n)} ,1 \le i \le n-1 \qquad (2)$$

Note that after the i^{th} column of this determinant (2), the corresponding columns are increased by one degree as compared with the ordinary Vandermonde determinant as in (1). In order to evaluate (2), let us construct an auxiliary determinant:

$$V_n^{(i)}(x) = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^i & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^i & \dots & x_2^n \\ : & & : & & : & & \\ 1 & x_n & x_n^2 & & x_n^i & & x_n^n \\ 1 & x & x^2 & \dots & x^i & \dots & x^n \end{vmatrix}_{(n+1)} ,1 \le i \le n \qquad (3)$$

Note that in (3), the i th column and a row involving the powers of x have been added, making a determinant of (n + 1) order.

Since $V_n^{(i)}(x)$ is now an ordinary Vandermonde determinant with indeterminates $x_1, x_2, \ldots, x_n$ and x, using the result in (1), we have :

$$V_n^{(i)}(x) = (x - x_1)(x - x_2)\cdots(x - x_n)\prod_{1\le p<q\le n}\left(x_q - x_p\right)$$

$$= \left[x^n - s_1x^{n-1} + s_2x^{n-2} - \cdots + (-1)^{n-i}s_{n-i}x^i + \cdots + (-1)^n s_n\right]\times V_n \qquad (4)$$

where $s_1, s_2, \ldots, s_n$ represent the elementary symmetric polynomials with indeterminates $x_1, x_2, \ldots, x_n$.

However, using Laplace expansion for the last row in (3), the x^i-term of $V_n^{(i)}(x)$ is exactly equal to $(-1)^{n+i}V_n^{(i)}$. By comparing coefficients with the x^i-terms in (4), we have :

$$(-1)^{n+i}V_n^{(i)} = (-1)^{n-i}s_{n-i}V_n$$

From this we therefore get :

$$V_n^{(i)} = s_{n-i}V_n = \left(\sum x_1x_2\cdots x_{n-i}\right)V_n \text{ for } 1 \le i \le n-1$$

and $\quad V_n^{(n)} = V_n \qquad (5)$

We now make use of the Vandermonde determinant to show that the system of equations :

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 0 & (6.1) \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 0 & (6.2) \\ \cdots\cdots & \\ x_1^n + x_2^n + \cdots + x_n^n = 0 & (6.n) \end{cases} \qquad (6)$$

has no non-trivial solution.

Rewrite (6) in the form :

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 0 \\ x_1x_1 + x_2x_2 + \cdots + x_nx_n = 0 \\ \cdots\cdots \\ x_1^{n-1}x_1 + x_2^{n-1}x_2 + \cdots + x_n^{n-1}x_n = 0 \end{cases} \qquad (7)$$

Note that the coefficient determinant of (7) is V_n. Therefore (7) has non-trivial solution iff

$$V_n = \prod_{1 \le p < q \le n} \left(x_q - x_p \right) = 0$$

i.e., without lost of generality, $x_n = x_{n-1}$.

But then the equations (6.2 – 6.n) can be written as :

$$\begin{cases} x_1{}^2 + x_2{}^2 + \cdots + x_{n-2}{}^2 + 2x_{n-1}{}^2 = 0 \\ x_1 x_1{}^2 + x_2 x_2{}^2 + \cdots + x_{n-2} x_{n-2}{}^2 + x_{n-1}\left(2x_{n-1}{}^2\right) = 0 \\ \\ x_1{}^{n-2} x_1{}^2 + x_2{}^{n-2} x_2{}^2 + \cdots + x_{n-2}{}^{n-2} x_{n-2}{}^2 + x_{n-1}{}^{n-2}\left(2x_{n-1}{}^2\right) = 0 \end{cases} \qquad (8)$$

(8) has non-trivial solution iff the coefficient determinant, i.e., $V_{n-1} = 0$, and we can get, without lost of generality, $x_{n-2} = x_{n-1}$. Continue in this way, using deduction (and hence induction), we can get $x_i = 0$ for i = 1 to n. As a result, (6) has no non-trivial solution.

Now, let us consider another system of equations :

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 0 \\ x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 = 0 \\ \cdots\cdots \\ x_1{}^{n-1} + x_2{}^{n-1} + \cdots + x_n{}^{n-1} = 0 \\ x_1{}^{n+1} + x_2{}^{n+1} + \cdots + x_n{}^{n+1} = 0 \end{cases} \quad (n \ge 2) \qquad (9)$$

Using the analogous layout as in the previous example, the coefficient determinant is exactly $V_n^{(n-1)}$. By (4), we have:

$$V_n{}^{(n-1)} = s_1 V_n = \left(x_1 + x_2 + \cdots + x_n\right) V_n = 0 \times V_n = 0$$

Therefore (9) must have non-trivial solutions. The reader may verify that the non-zero solution for (9) is:

$x_1 = k$, $x_2 = kw_n$, $x_3 = kw_n^2$, …., $x_n = kw_n^{n-1}$, where w_n is the n-th (complex) root of unity with $w_n^n = 1$, $1 + w_n + w_n^2 + \ldots + w_n^{n-1} = 0$.

Using the same principle, it is not difficult to prove that the system of equations :

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 0 \\ \quad\cdots\cdots \\ x_1^{n-2} + x_2^{n-2} + \cdots + x_n^{n-2} = 0 \quad (n \geq 3) \\ x_1^{n} + x_2^{n} + \cdots + x_n^{n} = 0 \\ x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} = 0 \end{cases} \tag{10}$$

has non-trivial solutions. The fact that $s_2 = \frac{1}{2}\left[\left(\sum x_i\right)^2 - \sum x_i^{\,2}\right]$ complete the proof.

Up to now, the readers may like to investigate the existence of non-trivial solutions in the general system :

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 0 \\ \quad\cdots\cdots \\ x_1^{i-1} + x_2^{i-1} + \cdots + x_n^{i-1} = 0 \\ x_1^{i+1} + x_2^{i+1} + \cdots + x_n^{i+1} = 0 \\ \quad\cdots\cdots \\ x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} = 0 \end{cases} \quad (1 \leq i \leq n) \tag{11}$$

Wish you can get some result. Good Luck!